Distribution & Interpolation Spaces – Solution sheet 1

Exercise 1. We first show that $f * g$ is continuous.
Let $x_n \to x$ in $\mathbb{R}^n$. Since $f, g \in C_c(\mathbb{R}^n)$, there exists a constant $M > 0$ such that $|f|, |g| \leq M$ for all $x \in \mathbb{R}^n$. Thus the sequence of functions $h_n(y) = f(y)g(x_n - y)$ satisfies

$$\begin{aligned} h_n(y) &\to f(y)g(x-y) \quad \forall y \in \mathbb{R}^n, \\ |h_n(y)| &\leq M^2 \chi_{\operatorname{spt} f}(y) \in L^1(\mathbb{R}^n), \end{aligned}$$

where $\chi_\Omega(y)$ denotes the characteristic function of the set Ω. Thus, by dominated convergence, we have

$$\lim_{x_n \to x} (f * g)(x_n) = \int_{\mathbb{R}^n} \lim_{x_n \to x} f(y)g(x_n - y)\, dy = \int_{\mathbb{R}^n} f(y)g(x-y)\, dy = (f * g)(x)\,,$$

which shows the continuity of $f * g$.

Now, if $x \notin \operatorname{spt} f + \operatorname{spt} g$, we have that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\, dy = \int_{\operatorname{spt} f} f(y)g(x-y)\, dy = 0,$$

since $g(x-y) = 0$ for all $y \in \operatorname{spt} f$ by our choice of x. This proves

$$\operatorname{spt}(f * g) \subset \operatorname{spt} f + \operatorname{spt} g.$$

To conclude we need still to observe that $f * g \in L^1(\mathbb{R}^n)$ and satisfies the desired estimate. Since we showed that $f * g$ is continuous and with compact support, then $f * g$ is bounded and bounded measurable functions with compact support belong to $L^1(\mathbb{R}^n)$.
By Tonelli theorem

$$\begin{aligned} \|f * g\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y)g(x-y) dy \right| dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)g(x-y)|\, dy dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)g(x-y)|\, dx dy \\ &= \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)| dx dy \\ &= \int_{\mathbb{R}^n} |f(y)| \|g\|_{L^1(\mathbb{R}^n)} dy \\ &= \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Exercise 2. We have that $|f_k| \leq |f|$ and $f_k(x) \to f(x)$ for almost every $x \in \mathbb{R}^n$. Thus by dominated convergence

$$f_k \to f \; in \; L^1(\mathbb{R}^n)\,.$$

The functions f_k are bounded by construction, moreover

$$|f_k * g(x)| \leq \int_{\mathbb{R}^n} |f_k(y)g(x-y)|dy \leq \|f_k\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} \leq k\|g\|_{L^1(\mathbb{R}^n)}$$

and

$$\int_{\mathbb{R}^n} |f_k * g(x)|dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f_k(y)||g(x-y)|dydx = \|f_k\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)},$$

thus ensuring that $f_k * g \in L^1 \cap L^\infty(\mathbb{R}^n)$ for every k. Note that, also, $f * g \in L^1(\mathbb{R}^n)$ since

$$\begin{aligned}\int_{\mathbb{R}^n} |(f*g)(x)|\,dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)||g(x-y)|\,dydx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)||g(x-y)|\,dxdy \\ &= \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)|\,dxdy = \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} < \infty\,.\end{aligned}$$

By direct computation we then conclude

$$\begin{aligned}\int_{\mathbb{R}^n} |(f_k*g)(x) - (f*g)(x)|\,dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f_k(y) - f(y)||g(x-y)|\,dydx \\ &\leq \|g\|_{L^1(\mathbb{R}^n)}\|f_k - f\|_{L^1(\mathbb{R}^n)} \to 0\,.\end{aligned}$$

Exercise 3.

i) Let

$$\begin{aligned}H : \mathbb{R}^n \times \mathbb{R}^n &\to [-\infty, +\infty] \\ (x,y) &\mapsto f(x-y)g(y)\end{aligned}$$

Since f, g are measurable, then also H is measurable. Following the computations already done in Exercise 1, we show that $H \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$: by Tonelli theorem

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} |H(x,y)|d(x\times y) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|dx\right)dy \\ &= \int_{\mathbb{R}^n} |g(y)|\left(\int_{\mathbb{R}^n} |f(x-y)|dx\right)dy \\ &= \int_{\mathbb{R}^n} |g(y)|\|f\|_{L^1(\mathbb{R}^n)}dy \\ &= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} \\ &< \infty.\end{aligned}$$

Hence, by Fubini theorem, for almost every $x \in \mathbb{R}^n$, the section

$$\begin{aligned}H_x : \mathbb{R}^n &\to [-\infty, +\infty] \\ y &\mapsto H(x,y) = f(x-y)g(y)\end{aligned}$$

is integrable, and the function

$$\begin{aligned} I_H &: \mathbb{R}^n \to \mathbb{R} \\ x \mapsto I_H(x) &= \begin{cases} \int_{\mathbb{R}^n} H_x(y)dy & \text{if } H_x \text{ is integrable,} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

belongs to $L^1(\mathbb{R}^n)$. Therefore we define $f*g$ to be (the equivalence class) of the function I_H in $L^1(\mathbb{R}^n)$. The estimate follows again by Fubini theorem and the previous computation:

$$\begin{aligned} \|f*g\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} H_x(y)dy \right| dx \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |H_x(y)|dydx \\ &= \|H\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \\ &= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

ii) If $r=\infty$, then it becomes the usual Hölder inequality, indeed

$$|(f*g)(x)| \leq \int_{\mathbb{R}^n} |f(y)||g(x-y)|\,dy \leq \|f\|_{L^p(\mathbb{R}^n)}\|g(x-\cdot)\|_{L^{p'}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)},$$

where $\|g(x-\cdot)\|_{L^{p'}(\mathbb{R}^n)}$ means that the norm is computed in the y variable, for a fixed $x\in\mathbb{R}^n$.

If $p=\infty$ (the case $p'=\infty$ is analogous) then we need to have $p'=1$ and consequently $r=\infty$. Thus the proof runs as in the previous case.

Suppose now that $r,p,p'<\infty$. We rewrite

$$f(y)g(x-y) = \left(f(y)^p g(x-y)^{p'}\right)^{\frac{1}{r}} f(y)^{1-\frac{p}{r}} g(x-y)^{1-\frac{p'}{r}}.$$

Note that

$$\frac{1}{r} + \frac{1}{\frac{p}{1-\frac{p}{r}}} + \frac{1}{\frac{p'}{1-\frac{p'}{r}}} = \frac{1}{r} + \frac{r-p}{rp} + \frac{r-p'}{rp'} = \frac{1}{p} + \frac{1}{p'} - \frac{1}{r} = 1\,.$$

Thus, by using the Hölder inequality with 3 indexes, we have

$$\int_{\mathbb{R}^n} f(y)g(x-y)\,dy \leq \left(\int_{\mathbb{R}^n} g(x-y)^{p'} f(y)^p\,dy\right)^{\frac{1}{r}} \|f\|^{1-\frac{p}{r}}_{L^p(\mathbb{R}^n)} \|g\|^{1-\frac{p'}{r}}_{L^{p'}(\mathbb{R}^n)}\,.$$

By taking the L^r-norm on both sides we get

$$\|f*g\|_{L^r(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)^{p'} f(y)^p\,dydx\right)^{\frac{1}{r}} \|f\|^{1-\frac{p}{r}}_{L^p(\mathbb{R}^n)} \|g\|^{1-\frac{p'}{r}}_{L^{p'}(\mathbb{R}^n)}\,. \tag{1}$$

Note that by Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)^{p'} f(y)^p\,dydx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)^{p'} f(y)^p\,dxdy \\ &= \int_{\mathbb{R}^n}\left(f(y)^p \int_{\mathbb{R}^n} g(x-y)^{p'}\,dx\right)dy \\ &= \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\,. \end{aligned} \tag{2}$$

Finally, by plugging identity (2) in the inequality (1), we conclude

$$\|f*g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{r}} \|g\|_{L^{p'}(\mathbb{R}^n)}^{\frac{p'}{r}} \|f\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{r}} \|g\|_{L^{p'}(\mathbb{R}^n)}^{1-\frac{p'}{r}} = \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}\,.$$

Exercise 4. *Step 1.* Let us prove first the statement in the simpler setting where g has compact support. Let us denote by $K \subset \mathbb{R}^n$ the support of g and let x be an arbitrary point in $\mathbb{R}^n$. In order to prove that $f*g$ is continuous in x, let

$$B(x-K,1) = \{z \in \mathbb{R}^n : \exists y \in K : |(x-y)-z| \le 1\}\,.$$

Since K is compact, then also $B(x-K,1)$ is compact and therefore the restriction of f to $B(x-K,1)$ is uniformly continuous, namely $\forall \varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$|x'-x''| \le \delta \text{ with } x', x'' \in B(x-K,1) \quad \Rightarrow \quad |f(x')-f(x'')| < \varepsilon.$$

Let $x, \tilde{x} \in \mathbb{R}^n$ with $|x-\tilde{x}| \le \delta$. Then

$$\begin{aligned} |f*g(x) - f*g(\tilde{x})| &= \left| \int_{\mathbb{R}^n} (f(x-y) - f(\tilde{x}-y)) g(y) dy \right| \\ &\le \int_{\mathbb{R}^n} |f(x-y) - f(\tilde{x}-y)||g(y)| dy \\ &= \int_K |f(x-y) - f(\tilde{x}-y)||g(y)| dy. \end{aligned}$$

Since for every $y \in K$ we have the both $x' = x-y$ and $x'' = \tilde{x} - y$ belong to $B(x-K,1)$, we can estimate the last integral with $\varepsilon \int_K |g(y)| dy$. Since g belongs to $L^p(\mathbb{R}^n)$ and has compact support, then $g \in L^1(\mathbb{R}^n)$, therefore the obtained estimate

$$|f*g(x) - f*g(\tilde{x})| \le \varepsilon \int_K |g(y)| dy \qquad \forall \tilde{x} : |x-\tilde{x}| \le \delta$$

shows that $f*g$ is continuous in x.
We now prove the second part of the statement, again under the additional assumption that g has compact support. Exactly as before, we have that $\frac{\partial f}{\partial x_i}$ is uniformly continuous on the compact set $B(x-K,1)$, i.e. for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$|x'-x''| \le \delta \text{ with } x', x'' \in B(x-K,1) \quad \Rightarrow \quad \left| \frac{\partial f(x')}{\partial x_i} - \frac{\partial f(x'')}{\partial x_i} \right| < \varepsilon.$$

Let $\sigma \in (0,\delta)$. By the definition of $f*g$ and Lagrange theorem, we have

$$\begin{aligned} \frac{f*g(x+\sigma e_i) - f*g(x)}{\sigma} &= \int_{\mathbb{R}^n} g(y) \frac{f(x-y+\sigma e_i) - f(x-y)}{\sigma} dy \\ &= \int_{\mathbb{R}^n} g(y) \frac{\partial f}{\partial x_i}(x-y+t(\sigma)e_i) dy \end{aligned}$$

for some $t(\sigma) \in [0,\sigma]$ depending on y. Since $\sigma \in (0,\delta)$, by the uniform continuity of $\frac{\partial f}{\partial x_i}$ on $B(x-K,1)$, we have

$$\left| \frac{\partial f}{\partial x_i}(x-y+t(\sigma)e_i) - \frac{\partial f}{\partial x_i}(x-y) \right| < \varepsilon \qquad \forall y \in K.$$

Observe that

$$\begin{aligned}\left|\frac{f*g(x+\sigma e_i)-f*g(x)}{\sigma}-\frac{\partial f}{\partial x_i}*g(x)\right| &= \left|\int_{\mathbb{R}^n} g(y)\frac{\partial f}{\partial x_i}(x-y+t(\sigma)e_i)dy-\int_{\mathbb{R}^n} g(y)\frac{\partial f}{\partial x_i}(x-y)dy\right|\\ &\le \int_{\mathbb{R}^n}|g(y)|\left|\frac{\partial f}{\partial x_i}(x-y+t(\sigma)e_i)-\frac{\partial f}{\partial x_i}(x-y)\right|\\ &\le \varepsilon\|g\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, letting $\varepsilon\to 0^+$, we get

$$\frac{\partial(f*g)}{\partial x_i}(x)=\frac{\partial f}{\partial x_i}*g(x).$$

Moreover, $\frac{\partial f}{\partial x_i}$ is continuous, therefore by the first part of the exercise, $\frac{\partial f}{\partial x_i}*g$ is continuous and hence $\frac{\partial(f*g)}{\partial x_i}$ is continuous as well.
Repeating the argument for every $i=1,\dots,n$, this shows that $f*g\in C^1(\mathbb{R}^n)$.
Step 2. We remove the extra assumption that g has compact support.
For $\varepsilon>0$ fixed, there is a compact set $K_\varepsilon\subset\mathbb{R}^n$ and two functions $g_1,g_2\in L^p(\mathbb{R}^n)$ such that

$$g=g_1+g_2,\qquad \operatorname{spt}(g_1)\subset K_\varepsilon,\qquad \|g_2\|_{L^p(\mathbb{R}^n)}\le\varepsilon.$$

By Step 1 there exists $\delta\in(0,1)$ such that

$$|x-\tilde{x}|\le\delta\quad\Rightarrow\quad |f*g_1(x)-f*g_1(\tilde{x})|<\varepsilon\|g_1\|_{L^1(\mathbb{R}^n)},$$

then

$$\begin{aligned}|f*g(x)-f*g(\tilde{x})| &\le |f*g_1(x)-f*g_1(\tilde{x})|+|f*g_2(x)-f*g_2(\tilde{x})|\\ &\le \varepsilon\|g_1\|_{L^1(\mathbb{R}^n)}+2\|f*g_2\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

For every $x\in\mathbb{R}^n$, we have by Hölder inequality

$$|f*g_2(x)|\le\int_{\mathbb{R}^n}|g_2(y)|f(x-y)|dy\le\|g_2\|_{L^p(\mathbb{R}^n)}\|f\|_{L^{p'}(\mathbb{R}^n)}\le\varepsilon\|f\|_{L^{p'}(\mathbb{R}^n)},$$

hence $\|f*g_2\|_{L^\infty(\mathbb{R}^n)}\le\varepsilon\|f\|_{L^{p'}(\mathbb{R}^n)}$. Plugging this into the previous estimate, we get

$$|f*g(x)-f*g(\tilde{x})|\le\varepsilon\left(\|g_1\|_{L^1(\mathbb{R}^n)}+2\|f\|_{L^{p'}(\mathbb{R}^n)}\right),$$

which proves the continuity of $f*g$.
The same argument works also for $\frac{\partial f}{\partial x_i}$: in the same way we get

$$\left\|\frac{f*g_2(x+\sigma e_i)-f*g_2(x)}{\sigma}\right\|_{L^\infty(\mathbb{R}^n)}\le\varepsilon\left\|\frac{\partial f}{\partial x_i}\right\|_{L^{p'}(\mathbb{R}^n)}$$

and then

$$\begin{aligned}\left|\frac{f*g(x+\sigma e_i)-f*g(x)}{\sigma}-\frac{\partial f}{\partial x_i}*g(x)\right| &\le \left|\frac{f*g_1(x+\sigma e_i)-f*g_1(x)}{\sigma}-\frac{\partial f}{\partial x_i}*g_1(x)\right|\\ &\quad+\left|\frac{f*g_2(x+\sigma e_i)-f*g_2(x)}{\sigma}\right|+\left|\frac{\partial f}{\partial x_i}*g_2(x)\right|\\ &\le\varepsilon\left(\|g_1\|_{L^p(\mathbb{R}^n)}+2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^{p'}(\mathbb{R}^n)}\right).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\frac{\partial(f*g)}{\partial x_i}(x) = \frac{\partial f}{\partial x_i} * g(x).$$

Since $\frac{\partial f}{\partial x_i} * g$ is continuous by the previous point, we have in particular that $f * g \in C^1(\mathbb{R}^n)$.

Exercise 5.

i) Let $\epsilon > 0$. Choose $\delta > 0$ with the following property: $\forall x, y$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ (note that this δ exists since f is uniformly continuous).

   Then, for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned} |f(x) - (f*\rho_\delta)(x)| &= \left| f(x) - \int_{B(0,\delta)} f(x-y)\rho_\delta(y)\,dy \right| \\ &= \left| \int_{B(0,\delta)} \big(f(x) - f(x-y)\big)\rho_\delta(y)\,dy \right| \\ &\le \int_{B(0,\delta)} |f(x) - f(x-y)|\rho_\delta(y)\,dy \le \epsilon\,. \end{aligned}$$

   So, we proved that $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\operatorname*{Sup}_{x\in\mathbb{R}^n} |f(x) - (f*\rho_\delta)(x)| < \epsilon\,,$$

   which means that $f * \rho_\delta \to f$ uniformly, as $\delta \to 0$.

ii) The idea is to apply Point i) to uniformly continuous functions approaching f.

   Step 1. Let $\varepsilon > 0$ and let us build a uniformly continuous function $\tilde{\phi}_\varepsilon$ such that $\|f - \tilde{\phi}_\varepsilon\|_{L^p(\mathbb{R}^n)} < \varepsilon$. Since $f \in L^p(\mathbb{R}^n)$, then there exists a step function

$$\phi_\varepsilon = \sum_{i=1}^{h} c_i \chi_{K_i} \qquad \text{such that } \|f - \phi_\varepsilon\|_{L^p(\mathbb{R}^n)} < \frac{\varepsilon}{2},$$

   where $h \in \mathbb{N}$, $c_i \in \mathbb{R}$, χ_{K_i} denotes the characteristic function of K_i and the sets $K_i \subset \mathbb{R}^n$ are compact and pairwise disjoint. In order to build a continuous approximation, observe that given a compact set K and $c \in \mathbb{R}$ the functions

$$\nu_M : x \mapsto c\,(1 - M\mathrm{dist}(x,K))^+\,, \qquad \text{with } M \in \mathbb{R}$$

   are uniformly continuous with compact support and they converge to $c\chi_K$ in $L^p(\mathbb{R}^n)$ for every $p \in [1, +\infty)$ as $M \to +\infty$. Here we used the notation $(z)^+ = \max\{z, 0\}$ to denote the positive part of z. By approximating each of the h multiples of the characteristic functions in the definition of ϕ_ε, we obtain a uniformly continuous function $\tilde{\phi}_\varepsilon$ with compact support that, for M sufficiently large, satisfies

$$\|\phi_\varepsilon - \tilde{\phi}_\varepsilon\|_{L^p(\mathbb{R}^n)} < \frac{\varepsilon}{2}.$$

   In particular $\|\phi_\varepsilon - f\|_{L^p(\mathbb{R}^n)} < \varepsilon$ and this concludes Step 1.

Step 2. By Point i) we have that

$$\tilde{\phi}_\varepsilon * \rho_\delta \to \tilde{\phi}_\varepsilon \qquad \text{uniformly as } \delta \to 0.$$

In particular there exists $\bar{\delta} > 0$ such that for every $\delta \in (0, \delta)$ one has

$$\|\tilde{\phi}_\varepsilon - \tilde{\phi}_\varepsilon * \rho_\delta\|_{L^p(\mathbb{R}^n)} < \varepsilon.$$

Therefore, for $\delta \in (0, \delta)$, we have

$$\begin{aligned} \|f - f * \rho_\delta\|_{L^p(\mathbb{R}^n)} &\leq \|f - \tilde{\phi}_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|\tilde{\phi}_\varepsilon - \tilde{\phi}_\varepsilon * \rho_\delta\|_{L^p(\mathbb{R}^n)} + \|\tilde{\phi}_\varepsilon * \rho_\delta - f * \rho_\delta\|_{L^p(\mathbb{R}^n)} \\ &< 3\varepsilon, \end{aligned}$$

where in the last inequality we used

$$\|\tilde{\phi}_\varepsilon * \rho_\delta - f * \rho_\delta\|_{L^p(\mathbb{R}^n)} \leq \|f - \tilde{\phi}_\varepsilon\|_{L^p(\mathbb{R}^n)} \|\rho_\delta\|_{L^1(\mathbb{R}^n)} < \varepsilon \cdot 1.$$

This shows that $f * \rho_\delta \to f$ in $L^p(\mathbb{R}^n)$.

iii) Given $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$, we show that there exists a function $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^p(\mathbb{R}^n)} < 2\varepsilon$.

Observe that a natural candidate would be $g = f * \rho_\delta$ with δ small enough but this choice does not have compact support. Instead, we can consider $g = \tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)}$, where $\delta(\varepsilon) > 0$ is such that $\|\tilde{\phi}_\varepsilon - \tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)}\|_{L^p(\mathbb{R}^n)} < \varepsilon$. In particular

$$\|f - \tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)}\|_{L^p(\mathbb{R}^n)} \leq \|f - \tilde{\phi}_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|\tilde{\phi}_\varepsilon - \tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)}\|_{L^p(\mathbb{R}^n)} < 2\varepsilon.$$

Notice moreover that $\tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)}$ has compact support since both $\tilde{\phi}_\varepsilon$ and $\rho_{\delta(\varepsilon)}$ have compact support and $\tilde{\phi}_\varepsilon * \rho_{\delta(\varepsilon)} \in C^\infty(\mathbb{R}^n)$ since $\rho_{\delta(\varepsilon)} \in C^\infty(\mathbb{R}^n)$.